hep-th/9612023

A Nonlocal Transcendental Realization of the Sugawara Operators at Arbitrary Level*

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Abstract

A recently found new free field realization of the affine Sugawara operators at arbitrary level is reviewed, which involves exponentials of the well-known DDF operators in string theory.

1 Results

The Sugawara operators play an important role in the representation theory of affine Lie algebras (see e.g. [1]). Using a string vertex operator construction of the affine algebra at arbitrary level it is possible to find a new free field realization of the Sugawara operators at arbitrary level in terms of physical string "DDF oscillators". This article reviews these results which were obtained in collaboration with K. Koepsell and H. Nicolai. A complete set of references can be found in [2].

Let $\bar{\mathfrak{g}}$ be a finite-dimensional simple Lie algebra of type ADE and rank $d-2$ ($d \geq 3$). Consider the associated nontwisted affine Lie algebra $\mathfrak{g}$ of rank $d-1$ with Cartan–Weyl basis H_n^i, $E_n^{\mathbf{r}}$ ($1 \leq i \leq d-2$, $\mathbf{r} \in \bar{\Delta}$, $n \in \mathbb{Z}$). The Sugawara operators

$$\mathcal{L}_m := \frac{1}{2(\ell + h^\vee)} \sum_{n\in\mathbb{Z}} \Big(\sum_{i=1}^{d-2} {}^\times_\times H_n^i H_{m-n}^i {}^\times_\times + \sum_{\mathbf{r}\in\bar{\Delta}} {}^\times_\times E_n^{\mathbf{r}} E_{m-n}^{-\mathbf{r}} {}^\times_\times \Big) \tag{1}$$

then form a Virasoro algebra with central charge $c_\ell := \frac{\ell \dim \bar{\mathfrak{g}}}{\ell + h^\vee}$, where ℓ and $h^\vee$ denote the level and the dual Coxeter number, respectively. Our main result is that in a certain string model the operators can be rewritten in terms of free oscillators A_m^i:

$$\mathcal{L}_m \;=\; \frac{1}{2\ell} \sum_{n\in\mathbb{Z}} \sum_{i=1}^{d-2} {}^\times_\times A_{\ell n}^i A_{\ell(m-n)}^i {}^\times_\times + \frac{h^\vee}{2\ell(\ell + h^\vee)} \sum_{n\neq 0(\ell)} \sum_{i=1}^{d-2} {}^\times_\times A_n^i A_{\ell m-n}^i {}^\times_\times$$

*Talk presented at *XXI International Colloquium on Group Theoretical Methods in Physics, 15 - 20 July 1996, Goslar, Germany*

†Supported by Deutsche Forschungsgemeinschaft

$$- \frac{1}{2\ell(\ell+h^\vee)} \sum_{\mathbf{r}\in\bar{\Delta}} \sum_{p=1}^{\ell-1} \frac{1}{|\zeta^p - 1|^2} \oint \frac{dz}{2\pi i}\, z^{\ell m-1} {}^\times_\times \mathrm{e}^{i\mathbf{r}\cdot[\mathcal{X}(z_p)-\mathcal{X}(z)]} {}^\times_\times$$

$$+ \frac{(\ell^2-1)(d-2)h^\vee}{24\ell(\ell+h^\vee)}\delta_{m,0}, \tag{2}$$

where $\zeta := \mathrm{e}^{2\pi i/\ell}$, $z_p := \zeta^p z$, and

$$\mathcal{X}^i(z) := Q^i - iA_0^i \ln z + i\sum_{m\neq 0} \frac{1}{m} A_m^i z^{-m}. \tag{3}$$

(The zero modes Q^i apparently drop out in (2) but will be needed later.) It will turn out that the "DDF oscillators" A_m^i are constructed from exponentials of the ordinary string oscillators α_n^ν. Therefore the $\mathcal{L}_m$'s are in fact "doubly transcendental" functions of the string oscillator modes. It is easy to see that for the special case $\ell = 1$ the Sugawara operators take the well-known form

$$\mathcal{L}_m = \frac{1}{2}\sum_{n\in\mathbb{Z}} \sum_{i=1}^{d-2} {}^\times_\times H_n^i H_{m-n}^i {}^\times_\times, \tag{4}$$

which is nothing but the equivalence of the Virasoro and the Sugawara construction at level 1 proved by I. Frenkel. On the other hand, if we consider the action of the zero mode operator on an arbitrary level-ℓ affine highest weight vector $|\mathbf{\Lambda}\rangle$, then we obtain (after invoking some identity for sums over roots of unity) the result

$$\mathcal{L}_0|\mathbf{\Lambda}\rangle = \frac{(\bar{\mathbf{\Lambda}} + 2\bar{\boldsymbol{\rho}})\cdot\bar{\mathbf{\Lambda}}}{2(\ell+h^\vee)}|\mathbf{\Lambda}\rangle, \tag{5}$$

which had previously been derived by exploiting the properties of the affine Casimir operator (see e.g. [1]), whereas here it can be simply read off from the general formula as a special case. Finally, formula (2) exhibits some nonlocal structure due to a new feature in the operator product expansion. In conformal field theory the singular part of the operator product expansion usually involves negative powers of $z - w$ leading to poles at $z = w$. In our case, however, we will demonstrate the appearance of negative powers of $z^\ell - w^\ell$ which produces poles at $z = w_p := \mathrm{e}^{2\pi i/\ell} w$ and is the origin of the nonlocal expressions in (2).

2 Compactified Bosonic String and DDF Construction

The string model in which we realize the affine Lie algebra and the Sugawara operators is "finite in all directions", i.e., we consider a (chiral half of a) closed bosonic string moving on a d-dim Minkowskian torus as spacetime such that the momentum lattice is given by the affine weight lattice Q^*. The usual string oscillators α_m^μ ($1 \leq \mu \leq$

d, $m \in \mathbb{Z}$) form a d-fold Heisenberg algebra $\mathbf{h}$, $[\alpha_m^\mu, \alpha_n^\nu] = m\eta^{\mu\nu}\delta_{m+n,0}$, and one introduces groundstates $|\boldsymbol{\lambda}\rangle = \mathrm{e}^{i\boldsymbol{\lambda}\cdot\mathbf{q}}|\mathbf{0}\rangle$ for $\boldsymbol{\lambda} \in Q^*$, which are by definition highest weight states for $\mathbf{h}$, i.e., $\alpha_0^\mu|\boldsymbol{\lambda}\rangle = \lambda^\mu|\boldsymbol{\lambda}\rangle$ and $\alpha_m^\mu|\boldsymbol{\lambda}\rangle = 0 \quad \forall m > 0$. ($q^\nu$ is position operator:$[q^\mu, p^\nu] = i\eta^{\mu\nu}$, $p^\mu \equiv \alpha_0^\mu$) Then the Fock space $\mathcal{F}$ is the direct sum of irreducible $\mathbf{h}$ modules: $\mathcal{F} = \mathrm{span}\{\alpha_{-m_1}^{\mu_1} \cdots \alpha_{-m_M}^{\mu_M}|\boldsymbol{\lambda}\rangle \,|\, 1 \le \mu_i \le d, m_i > 0, \boldsymbol{\lambda} \in Q^*\}$. To complete the quantization procedure one has to implement the physical state conditions. This amounts to restricing $\mathcal{F}$ to the subspace $\mathcal{P}$ of physical states, which are by definition conformal primary states of weight 1, viz.

$$\mathcal{P} := \bigoplus_{\boldsymbol{\lambda} \in Q^*} \mathcal{P}^{(\boldsymbol{\lambda})}, \qquad \mathcal{P}^{(\boldsymbol{\lambda})} := \{\psi \in \mathcal{F} \,|\, L_n\psi = \delta_{n0}\psi \;\forall n \ge 0,\; p^\mu\psi = \lambda^\mu\psi\}, \tag{6}$$

with respect to the Virasoro constraints (with $c = d$) $L_n := \frac{1}{2}\sum_{m\in\mathbb{Z}} :\boldsymbol{\alpha}_m\cdot\boldsymbol{\alpha}_{n-m}:$. One easily works out the simplest examples of physical string states to find tachyons $|\mathbf{a}\rangle$, satisfying $\mathbf{a}^2 = 2$, and photons $\boldsymbol{\xi}\cdot\boldsymbol{\alpha}_{-1}|\mathbf{k}\rangle$, satisfying $\boldsymbol{\xi}\cdot\mathbf{k} = \mathbf{k}^2 = 0$ ($\boldsymbol{\xi} \in \mathbb{R}^{d-1,1}$). This direct method quickly becomes rather cumbersome and one might ask whether there is an elegant way of describing physical states which also yields structural insights into $\mathcal{P}$. This is achieved by the so-called DDF construction.

Let us consider a fixed momentum vector $\boldsymbol{\lambda} \in Q^*$ satisfying $\boldsymbol{\lambda}^2 \le 2$ (otherwise it could not give rise to physical states). In order to find a complete basis for $\mathcal{P}^{(\boldsymbol{\lambda})}$, one starts from a so-called DDF decomposition of $\boldsymbol{\lambda}$,

$$\boldsymbol{\lambda} = \mathbf{a} - n\mathbf{k}, \qquad n = 1 - \tfrac{1}{2}\boldsymbol{\lambda}^2 \;, \tag{7}$$

for some tachyon $|\mathbf{a}\rangle$ and lightlike vector $\mathbf{k}$ satisfying $\mathbf{a}\cdot\mathbf{k} = 1$. Such a decomposition is always possible although neither $\mathbf{a}$ nor $\mathbf{k}$ will in general lie on the affine weight lattice. Next we choose $d-2$ orthonormal polarization vectors $\boldsymbol{\xi}^i \in \mathbb{R}^{d-1,1}$ such that $\boldsymbol{\xi}^i\cdot\mathbf{a} = \boldsymbol{\xi}^i\cdot\mathbf{k} = 0$. The transversal DDF operators are now defined as

$$A_m^i(\mathbf{a},\mathbf{k}) := \oint \frac{dz}{2\pi i}\, \boldsymbol{\xi}^i\cdot\mathbf{P}(z)\, \mathrm{e}^{im\mathbf{k}\cdot\mathbf{X}(z)}, \tag{8}$$

with the Fubini–Veneziano coordinate and momentum fields respectively given by

$$X^\mu(z) := q^\mu - ip^\mu \ln z + i\sum_{m\neq 0} \frac{1}{m}\alpha_m^\mu z^{-m}, \qquad P^\mu(z) := \sum_{m\in\mathbb{Z}} \alpha_m^\mu z^{-m-1}. \tag{9}$$

It is straightforward to show that the A_m^i's realize a $(d-2)$-fold "transversal" Heisenberg algebra, $[A_m^i, A_n^j] = m\delta^{ij}\delta_{m+n,0}$. One also introduces longitudinal DDF operators $A_m^-(\mathbf{a},\mathbf{k})$, which are much more complicated expressions (and whose explicit form is not needed here). They form a "longitudinal" Virasoro algebra ($c = 26 - d$) and satisfy $[A_n^i, A_m^-] = 0$. It turns out that the DDF operators constitute a spectrum-generating algebra for $\mathcal{P}^{(\boldsymbol{\lambda})}$ in the following sense. First, one can show that $[A_n^i, L_m] = [A_n^-, L_m] = 0 \;\forall m$, which implies that they map physical states into physical states.

Furthermore, the tachyonic state $|\mathbf{a}\rangle$ is annihilated by the DDF operators with non-negative mode index. Finally, the DDF operators provide a basis for $\mathcal{P}^{(\boldsymbol{\lambda})}$, viz.

$$\mathcal{P}^{(\boldsymbol{\lambda})} = \mathrm{span}\{A^{i_1}_{-n_1}\cdots A^{i_N}_{-n_N}A^-_{-m_1}\cdots A^-_{-m_M}|\mathbf{a}\rangle \,|\, n_1+\ldots+m_M = 1-\tfrac{1}{2}\boldsymbol{\lambda}^2\}. \quad (10)$$

3 Realization of Affine Lie Algebra

We will now employ a vertex operator construction for the affine Lie algebra at arbitrary level to find an explicit realization of the Sugawara operators in terms of the string oscillators. Let $L(\boldsymbol{\Lambda})$ denote an irreducible level-ℓ affine highest weight module with vacuum vector $v_{\boldsymbol{\Lambda}}$, dominant integral weight $\boldsymbol{\Lambda}\in Q^*$, and weight system $\Omega(\boldsymbol{\Lambda})$. Without loss of generality we may assume that $\boldsymbol{\Lambda}^2=2$ (due to $L(\boldsymbol{\Lambda})\cong L(\boldsymbol{\Lambda}+z\boldsymbol{\delta})$ for $z\in\mathbb{C}$ and the affine null root $\boldsymbol{\delta}$). We define a Cartan–Weyl basis for $\mathfrak{g}$ by

$$H^i_m := \oint\frac{dz}{2\pi i}\,\boldsymbol{\xi}^i\cdot\mathbf{P}(z)\,\mathrm{e}^{im\boldsymbol{\delta}\cdot\mathbf{X}(z)}, \qquad E^{\mathbf{r}}_m := \oint\frac{dz}{2\pi i}\,{:}\mathrm{e}^{i(\mathbf{r}+m\boldsymbol{\delta})\cdot\mathbf{X}(z)}{:}\,c_{\mathbf{r}}, \quad (11)$$

where $c_{\mathbf{r}}$ is some cocycle factor. The central element and the exterior derivative are given by $K:=\boldsymbol{\delta}\cdot\mathbf{p}$ and $d:=\boldsymbol{\Lambda}_0\cdot\mathbf{p}$, respectively. This yields a level-ℓ vertex operator realization of $\mathfrak{g}$ on the space of physical states with $v_{\boldsymbol{\Lambda}}\equiv|\boldsymbol{\Lambda}\rangle$, viz.

$$L(\boldsymbol{\Lambda})\hookrightarrow\mathcal{P}(\boldsymbol{\Lambda}) := \bigoplus_{\boldsymbol{\lambda}\in\Omega(\boldsymbol{\Lambda})}\mathcal{P}^{(\boldsymbol{\lambda})}, \qquad L(\boldsymbol{\Lambda})_{\boldsymbol{\lambda}}\hookrightarrow\mathcal{P}^{(\boldsymbol{\lambda})}. \quad (12)$$

Note that the operators $A^i_{\ell m}\equiv H^i_m$ are not only part of the transversal Heisenberg algebra but also make up the homogeneous Heisenberg subalgebra of $\mathfrak{g}$. Only at level 1 these two algebras coincide. It is intriguing to see how in the above construction both the vacuum vector conditions $e_I|\boldsymbol{\Lambda}\rangle=0$ and the null vector conditions $f_I^{1+\mathbf{r}_I\cdot\boldsymbol{\Lambda}}|\boldsymbol{\Lambda}\rangle=0$ for $0\le I\le d-2$ (in terms of the affine Chevalley generators e_I, f_I) immediately follow from the physical state condition. Below, we will see that only transversal physical states can occur in the affine highest weight module $L(\boldsymbol{\Lambda})$. Hence we effectively deal with the embedding $L(\boldsymbol{\Lambda})_{\boldsymbol{\lambda}}\hookrightarrow\mathcal{P}^{(\boldsymbol{\lambda})}_{\text{transv.}}$ and have the following universal estimate for affine weight multiplicities at arbitrary level:

$$\mathrm{mult}_{\boldsymbol{\Lambda}}(\boldsymbol{\lambda})\equiv\dim L(\boldsymbol{\Lambda})_{\boldsymbol{\lambda}}\le\dim\mathcal{P}^{(\boldsymbol{\lambda})}_{\text{transv.}} = p_{d-2}(1-\tfrac{1}{2}\boldsymbol{\lambda}^2), \quad (13)$$

where $p_{d-2}(n)$ counts the partition of n into "parts" of $d-2$ "colours".

Finally, we would like to sketch how the new formula (2) is obtained. If we insert the above expressions for the step operators $E^{\mathbf{r}}_n$ into (1), we encounter in $\sum_{n\in\mathbb{Z}}{}^{\times}_{\times}E^{\mathbf{r}}_nE^{-\mathbf{r}}_{m-n}{}^{\times}_{\times}$ an operator-valued infinite series

$$Y(z,w) := \sum_{n\ge0}\mathrm{e}^{in\boldsymbol{\delta}\cdot[\mathbf{X}(z)-\mathbf{X}(w)]}. \quad (14)$$

It is not difficult to see that application to the groundstate $|\mathbf{a}\rangle$ yields

$$Y(z,w)|\mathbf{a}\rangle = \sum_{n\geq 0}\left[\left(\frac{z}{w}\right)^{\ell n} + \text{osc. terms}\right]|\mathbf{a}\rangle = \frac{z^\ell}{z^\ell - w^\ell}|\mathbf{a}\rangle + \text{excited states}, \qquad (15)$$

which explicates the origin of the nonlocalities. Another important point to notice is that the Sugawara operators are physical by construction. Consequently, it must be possible to rewrite them in a manifestly physical form, i.e., in terms of the DDF oscillators. At first sight this seems to be a hopeless task but there is a little trick. Motivated by the observation that the leading oscillator contribution in a DDF operator A^i_m is given by $\boldsymbol{\xi}^i\cdot\boldsymbol{\alpha}_m$, one simply replaces every oscillator term $\mathbf{r}\cdot\boldsymbol{\alpha}_m$ occurring in the vertex operator realization of the Sugawara operators by a "DDF oscillator" $(\mathbf{r}\cdot\boldsymbol{\xi}^i)A^i_m$. The resulting formula (2) then indeed gives an alternative expression for the Sugawara operators in terms of the transversal DDF operators [2].

This method also works for the step operators themselves [3]. One ends up with

$$E^{\mathbf{r}}_m\Big|_{\mathcal{P}(\boldsymbol{\Lambda})} = \oint\frac{dz}{2\pi i}\, z^{\ell m}{}_\times^\times e^{i\mathbf{r}\cdot\mathcal{X}(z)}{}_\times^\times c_{\mathbf{r}}, \qquad (16)$$

with the new transversal coordinate field $\mathcal{X}(z)$ given in (3). There is one subtlety here concerning the zero modes. Since the position operators q^μ are unphysical we have to replace them by some physical analogue. What should that be? The correct answer involves the (physical!) Lorentz generators $M^{\mu\nu}$, viz.

$$Q^i := (\boldsymbol{\xi}^i)_\mu(\mathbf{k}_\ell)_\nu M^{\mu\nu}. \qquad (17)$$

One shows that $e^{i\mathbf{r}\cdot\mathbf{Q}}\mathbf{v}\cdot\boldsymbol{\alpha}_m e^{-i\mathbf{r}\cdot\mathbf{Q}} = {}^{[\ell]}t_{\mathbf{r}}(\mathbf{v})\cdot\boldsymbol{\alpha}_m$, where ${}^{[\ell]}t_{\mathbf{r}}(\mathbf{v}) := \mathbf{v} + (\mathbf{v}\cdot\mathbf{k}_\ell)\mathbf{r} - \left[(\mathbf{v}\cdot\mathbf{k}_\ell)\frac{1}{2}\mathbf{r}^2 + \mathbf{r}\cdot\mathbf{v}\right]\mathbf{k}_\ell$ for $\mathbf{v}\in\mathfrak{h}^*$, $\mathbf{k}_\ell := \frac{1}{\ell}\boldsymbol{\delta}$, $\mathbf{r}\in\bar{Q}$. Hence the resolution of the zero mode problem provides us with a realization of the well-known affine Weyl translations ${}^{[\ell]}t_{\mathbf{r}}$ in terms of Lorentz boosts.

How will the longitudinal DDF operators fit into this framework? Clearly, the full space of physical states is an infinite direct sum of irreducible $\mathfrak{g}$-modules. Then the longitudinal DDF operators will map between different levels or, more generally, between different $\mathfrak{g}$-modules and will thus play the role of intertwining operators.

References

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